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A new proof of Zassenhaus theorem on finite groups of fixed-point-free automorphisms ☆

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Abstract

In 1935 H. Zassenhaus [Abh. Math. Semin. Hamb. Univ. 11 (1935) 187–220] stated that a non-trivial finite perfect group of fixed-point-free automorphisms of an abelian group is isomorphic to $SL_2(5)$. The paper contains a character-free proof of this theorem.

Keywords: Free action; Elements of order 3

1. Introduction

A group G acts *freely* (or, in other terminology, fixed-point-freely) on an additively written group V if $vg = v$ for $g \in G$, $v \in V$ only if $v = 0$ or $g = 1$. For example, Frobenius complement in a finite Frobenius group F acts freely on the core of F. All complements in finite Frobenius groups were classified in the famous paper of H. Zassenhaus [5]. The crucial point of this classification is the following result.

Zassenhaus Theorem. *If G is a perfect finite irreducible group of matrices of degree $d > 1$ over the complex number field $\mathbf{C}$ with the property that no non-trivial element of G has eigenvalue 1 then $d = 2$ and $G \simeq SL_2(5)$.*

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The original proof of this theorem (see also [3] and [6]) is complicated and depends heavily on the counting arguments and character theory of finite groups. The goal of this paper is to give an elementary proof based on the technique of quadratic elements for the prime 3 [4,7,8] and simple matrix calculations. We obtain Zassenhaus Theorem as a corollary of the following result.

Theorem 1.1. *Let G be a non-trivial finite irreducible group of linear transformations of a d-dimensional vector space V over $\mathbf{C}$. If G acts freely on V and is generated by elements of order* 3 *then $d \leqslant 2$ and either $|G| = 3$, or $G \simeq SL_2(3)$, or $G \simeq SL_2(5)$.*

2. Assumed results

Lemma 2.1. *Let P be a Sylow p-subgroup of a finite group G.*

(1) (O. Hölder, see [2], p. 420.) *If P is cyclic for every prime p dividing $|G|$ then G is soluble.*
(2) (W. Burnside, see [2], p. 419.) *If P lies in the center of $N_G(P)$ then G contains a normal p-complement.*
(3) (Well-known, see [2], p. 437.) *If $p = 2$, P is metacyclic and $|G|$ is co-prime to* 3 *then G contains a normal p-complement.*
(4) (Well-known, see [2], p. 310.) *If every abelian subgroup of P is cyclic then, for $p > 2$, P is cyclic and, for $p = 2$, P is metacyclic.*

Lemma 2.2.

(1) ([2], p. 140.) *The group $\langle x, y \mid x^2 = y^3 = (xy)^5 = 1\rangle$ is isomorphic to the alternating group A_5.*
(2) ([1], p. 172.) *The group $\langle x_1, \dots, x_n \mid x_i^3 = (x_i x_j)^2 = 1,\ i, j = 1, \dots, n,\ i \neq j\rangle$ is isomorphic to the alternating group A_{n+2}.*

The following lemma is a simple consequence of the well-known classification of the finite subgroups of $SL_2(\mathbf{C})$ (see, for example, Theorem XII.8.6 in [3]).

Lemma 2.3. *Let G be finite non-cyclic subgroup of $SL_2(\mathbf{C})$ generated by elements of order* 3*. Then G is isomorphic to $SL_2(3)$ or $SL_2(5)$.*

3. Proof of Theorem 1.1

Let V be a d-dimensional vector space over $\mathbf{C}$ and G a finite non-trivial group generated by elements of order 3. Suppose that G acts freely and irreducibly on V. We will consider G as a subgroup of the multiplicative group of the ring of all linear transformations of V.

Lemma 3.1.

(1) *If* $x, y \in G$ *are elements of order* 3 *and* 2, *respectively, then* $x^2 = x^{-1} = -1 - x$, $y = -1$.
(2) *If* $x, y \in G$ *are elements of order* 3 *then, for* $H = \langle x, y\rangle$, *there exists a non-trivial* H*-invariant subspace in* V *of dimension* 1 *or* 2 *and one of the following holds*:
  (i) $H = \langle x\rangle$;
  (ii) *the order of* xy *is* 4 *or* 6 *and* $H \simeq SL_2(3)$;
  (iii) *the order of* xy *is* 10 *and* $H \simeq SL_2(5)$.

Proof. If the order of x is 3 then, for every $v \in V$, $v(1 + x + x^2)x = v(1 + x + x^2)$ and, since G acts freely on V, $1 + x + x^2 = 0$. Similarly, if the order of y is 2 then $v(1 + y)y = v(1 + y)$, so $1 + y = 0$. This proves (1).

Suppose now that $x, y \in G$ are elements of order 3 such that $H = \langle x, y\rangle \neq \langle x\rangle$.

Replacing y by y^{-1} if necessary we can assume that $n \leqslant m$ where n is the order of xy and m is the order of xy^{-1}.

If $v \in V$ is an eigenvector for xy then $vxy = \lambda v$ where λ is a primitive nth root of unity. If v is an eigenvector for x then $H = \langle x, y\rangle$ is isomorphic to a subgroup of the multiplicative group of $\mathbf{C}$ and hence $\langle x, y\rangle = \langle x\rangle$ contrary to the assumption. Thus the subspace U spanned by $v_1 = v$, $v_2 = vx$ is two-dimensional and, by (1) of lemma,

$$v_1x = v_2, \qquad v_2x = vx^2 = -v - vx = -v_1 - v_2.$$

Furthermore, $vxy = \lambda v$, so $vx = \lambda vy^{-1} = -\lambda(vy + v)$, $vy = -v - \lambda^{-1}vx$, and hence

$$v_1y = -v_1 - \lambda^{-1}v_2, \qquad v_2y = \lambda v_1.$$

Thus, U is H-invariant and

$$x_1 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad y_1 = \begin{bmatrix} -1 & -\lambda^{-1} \\ \lambda & 0 \end{bmatrix}$$

are matrices of linear transformations in respect to the base v_1, v_2 of U which x, y, respectively, induce in U. Since H acts freely on U, the map $x \to x_1$, $y \to y_1$ can be extended to an isomorphism of H and $H_1 = \langle x_1, y_1\rangle$. It is obvious that $H_1 \leqslant SL_2(\mathbf{C})$. By Lemma 2.3, H is isomorphic to $SL_2(3)$ or $SL_2(5)$, and the rest of the lemma can be checked directly. $\square$

Lemma 3.2. *If* x, y, z *are elements of order* 3 *in* G *such that* $(xz)^2 = (yz)^2 = -1$, *then either* $\langle x, y\rangle = \langle x\rangle$ *and* $\langle x, y, z\rangle \simeq SL_2(3)$, *or* $\langle x, y, z\rangle = \langle x, y\rangle$, *or* $(xy)^2 = -1$ *and* $\langle x, y, z\rangle \simeq SL_2(5)$.

Proof. Let $H = \langle x, y, z\rangle$. If $L = \langle x, y\rangle$ is cyclic then, by (2) of Lemma 3.1, $L = \langle x\rangle$ and $H = \langle x, z\rangle$. In this case, $H/\langle -1\rangle$ is a homomorphic image of the group $\langle a, b \mid a^3 = b^3 = (ab)^2\rangle$ which, by (2) of Lemma 2.3, is isomorphic to A_4, so $H \simeq SL_2(3)$ by (2) of

Lemma 3.1. Suppose that L is non-cyclic. If $(xy)^2 = -1$ then $H/\langle -1\rangle \simeq A_5$ by (2) of Lemma 2.3 and since A_5 can be generated by two elements of order 3, $H \simeq SL_2(5)$ by Lemma 3.1. So, in this case, the conclusion is true.

Suppose $(xy)^2 \neq -1$. By (2) of Lemma 3.1, the order of xy is 6 or 10 and, by (1) of Lemma 3.1, $(xy)^s = -1$ where $s = 3$ or 5.

Let u be an eigenvector for xy in V, i.e., $u \neq 0$ and $u(xy) = \lambda u$, $\lambda \in \mathbf{C}$. Let $u_1 = u$, $u_2 = u_1x$, $u_3 = u_1z$, $u_4 = u_2z = u_1xz$. Notice that $u_1 \neq 0$ and

$$\lambda u_1 = \lambda u = u_1xy = u_2y. \tag{1}$$

By (1) of Lemma 3.1,

$$x^{-1} = x^2 = -1 - x, \quad y^{-1} = y^2 = -1 - y, \quad z^2 = z^{-1} = -1 - z. \tag{2}$$

Furthermore, the equality $(zx)^2 = -1$ gives

$$\begin{aligned} zx &= -(zx)^{-1} = -x^{-1}z^{-1} \\ &= -(-1-x)(-1-z) = -1 - x - z - xz. \end{aligned} \tag{3}$$

Similarly,

$$zy = -1 - y - z - yz. \tag{4}$$

By (2)–(4), $u_2x = u_1x^2 = u_1(-1-x) = -u_1 - u_1x = -u_1 - u_2$, $u_3x = u_1zx = u_1(-1-x-z-xz) = -u_1 - u_2 - u_3 - u_4$, $u_4x = u_2zx = u_1xzx = -u_1z^{-1} = -u_1(-1-z) = u_1 + u_1z = u_1 + u_3$.

By (1) and (2), $u_1x = \lambda u_1y^{-1} = \lambda u_1(-1-y) = -u_1 - u_1y$ which implies $u_1y = -u_1 - \lambda^{-1}u_2$. Furthermore, by (1), $u_2y = u_1xy = \lambda u_1$; by (4), $u_3y = u_1zy = u_1(-1-y-z-yz) = \lambda^{-1}u_1 + \lambda^{-1}u_1z = \lambda^{-1}u_1 + \lambda^{-1}u_2$, $u_4y = u_2zy = u_2(-1-y-z-yz) = -\lambda u_1 - u_2 - \lambda u_3 - u_4$; by (3), $u_3z = u_1z^2 = u_1(-1-z) = -u_1 - u_3$ and, similarly, $u_4z = -u_2 - u_4$.

Thus, H acts on $\langle u_1, u_2, u_3, u_4\rangle$ according to the following equalities:

$$\begin{aligned} &u_1x = u_2, \quad u_2x = -u_1 - u_2, \\ &u_3x = -u_1 - u_2 - u_3 - u_4, \quad u_4x = u_1 + u_3, \\ &u_1y = -u_1 - \lambda^{-1}u_2, \quad u_2y = \lambda u_1, \\ &u_3y = \lambda^{-1}(u_2 + u_4), \quad u_4y = -\lambda u_1 - u_2 - \lambda u_3 - u_4, \\ &u_1z = u_3, \quad u_2z = u_4, \quad u_3z = -u_1 - u_3, \quad u_4z = -u_2 - u_4. \end{aligned} \tag{5}$$

By direct calculation it is easy to check the following assertions:

Lemma 3.3.

(1) *Let* $x, y \in SL_2(\mathbf{C})$ *where*

$$x = \begin{bmatrix} . & 1 \\ -1 & -1 \end{bmatrix}, \qquad y = \begin{bmatrix} -1 & -\lambda^{-1} \\ \lambda & . \end{bmatrix}.$$

If λ *is a primitive* 6*th root of unity then*

$$x^{-1}y^{-1}x = \begin{bmatrix} -\lambda & . \\ \lambda^2 & \lambda^2 \end{bmatrix}.$$

If λ *is a primitive* 10*th root of unity then*

$$yxyx^{-1}y^{-1} = \begin{bmatrix} -\lambda^3+\lambda^4 & \lambda-\lambda^2+\lambda^3 \\ \lambda^3 & -\lambda+\lambda^2 \end{bmatrix}.$$

(2) *Let* $x, y, z \in SL_4(\mathbf{C})$ *where*

$$x = \begin{bmatrix} . & 1 & . & . \\ -1 & -1 & . & . \\ -1 & -1 & -1 & -1 \\ 1 & . & 1 & . \end{bmatrix}, \qquad y = \begin{bmatrix} -1 & -\lambda^{-1} & . & . \\ \lambda & . & . & . \\ . & \lambda^{-1} & . & \lambda^{-1} \\ -\lambda & -1 & -\lambda & -1 \end{bmatrix},$$

$$z = \begin{bmatrix} . & . & 1 & . \\ . & . & . & 1 \\ -1 & . & -1 & . \\ . & -1 & . & -1 \end{bmatrix}.$$

If λ *is a primitive* 6*th root of unity then* $vz = vxy^{-1}x^{-1}$ *where* $v = (\lambda, 0, 1, 0)$. *If* λ *is a primitive* 10*th root of unity then* $vz = vy^{-1}x^{-1}yxy$ *where* $v = (\lambda^3-\lambda^4, -\lambda+\lambda^2-\lambda^3, 1, 0)$.

Now, suppose that $s = 3$. By (5) and (2) of Lemma 3.3, $vz = vxy^{-1}x^{-1}$ where $v = \lambda u_1 + u_3$. If $z = xy^{-1}x^{-1}$ then $z \in \langle x, y\rangle$ and the lemma is proved. In contrary case, $v = 0$, i.e., $u_3 = -\lambda u_1$. By (5) and (1) of Lemma 3.1, $u_1 z = u_3 = u_1 x^{-1}y^{-1}x$ and, since $u_1 \neq 0$, $z \in \langle x, y\rangle$ again.

Suppose that $s = 5$. Again, by (5) and (2) of Lemma 3.3, $vz = vy^{-1}x^{-1}yxy$ where $v = (\lambda^3-\lambda^4)u_1 + (-\lambda+\lambda^2-\lambda^3)u_2 + u_3$. If $z = y^{-1}x^{-1}yxy = z$ then $z \in \langle x, y\rangle$ and the lemma is proved. In contrary case, $v = 0$, i.e., $u_3 = (-\lambda^3+\lambda^4)u_1 + (\lambda-\lambda^2+\lambda^3)u_2$. By (5) and (1) of Lemma 3.3, $u_1 z = u_3 = u_1 yxyx^{-1}y^{-1}$ and $z = yxyx^{-1}y^{-1} \in \langle x, y\rangle$. □

Let L be a subgroup of maximal order among subgroups of G generated by two elements x, y of order 3. By Lemma 3.1, it is suffices to prove that $L = G$.

Suppose that $L \neq G$. Then L is non-cyclic and there exist an element $z \in G \setminus L$ of order 3.

Suppose first, that $L \simeq SL_2(3)$, so G does not contain a subgroup isomorphic to $SL_2(5)$. Then $\langle x, z\rangle\rangle \simeq \langle y, z\rangle \simeq SL_2(3)$. Replacing, if necessary, z by z^{-1} and y by y^{-1}, we can assume that $(xy)^2 = (xz)^2 = -1$. If $(yz)^2 = -1$ then, by (2) of Lemma 2.3 and (2) of Lemma 3.1, $\langle x, y, z\rangle \simeq SL_2(5)$ contrary to the assumption. Hence $(yx)^2 \neq 1$, so, by (2) of Lemma 3.2, $x \in \langle y, z\rangle \simeq SL_2(3)$ and $\langle x, y, z\rangle = \langle x, y\rangle$ contrary to the choice of z.

Thus, $L \simeq SL_2(5)$ and hence $L/\langle -1\rangle \simeq A_5$. Let $\bar{a} = (123)$, $\bar{b} = (124)$, $\bar{c} = (125) \in A_5$. Then $(\bar{a}\bar{b})^2 = (\bar{a}\bar{c})^2 = (\bar{b}\bar{c})^2 = 1$. Let a, b, c be order 3 preimages in L of elements $\bar{a}, \bar{b}, \bar{c}$. Then a, b, c generate L and, by (1) of Lemma 3.1, $(ab)^2 = (ac)^2 = (bc)^2 = -1$.

If $\langle a, z\rangle \simeq SL_2(3)$ then, up to replacing z by z^{-1}, $(az)^2 = -1$. If additionally $(bz)^2 \neq -1$ then, by Lemma 3.2, $a \in \langle b, z\rangle$ and, since $z \notin \langle a, b\rangle$, $K = \langle a, b, z\rangle = \langle b, z\rangle \simeq SL_2(5)$. If $(bz)^2 = -1$ then, by (2) of Lemma 2.3 and (2) of Lemma 3.1, $K = \langle a, b, z\rangle \simeq SL_2(5)$.

Thus, in every case, G contains a subgroup $K \simeq SL_2(5)$ such that $K \neq L$ and $K \cap L$ contains a subgroup isomorphic to $SL_2(3)$. Since $|SL_2(5)|/|SL_2(3)| = 5$, $L \cap K \simeq SL_2(3)$. Choose in $L \cap K$ elements a, b of order 3 such that $(ab)^2 = -1$. Then there exists an element $c \in L$ of order 3 such that $(ac)^2 = (bc)^2 = -1$ and $L = \langle a, b, c\rangle$. Similarly, K contains an element d of order 3 such that $(ad)^2 = (bd)^2 = -1$ and $K = \langle a, b, d\rangle$. If $(cd)^2 \neq -1$ then, by Lemma 3.2, $x, y \in \langle c, d\rangle$ and hence $\langle L, K\rangle \leqslant \langle c, d\rangle$ which is impossible by (2) of Lemma 3.1. It follows that $(cd)^2 = -1$ and, by (2) of Lemma 2.2, $\langle a, b, c, d\rangle/\langle -1\rangle \simeq A_6$. Since A_6 contains a subgroup of order 9 generated by two elements of order 3, this contradicts (2) of Lemma 3.1.

This final contradiction proves Theorem 1.1.

4. Proof of Zassenhaus Theorem

By Theorem 1.1, it suffices to prove that a group G satisfying the conditions of Zassenhaus theorem is generated by elements of order 3.

Clearly, G acts freely on the underlying d-dimensional vector space V over $\mathbf{C}$, so if A is an abelian subgroup in G then, since V contains an one-dimensional A-invariant subspace, A is isomorphic to a subgroup of the multiplicative group of $\mathbf{C}$ and hence is cyclic. By (4) of Lemma 2.1, every Sylow p-subgroup P in G is cyclic for p odd and is metacyclic for $p = 2$.

If the order of G is co-prime to 3, then G contains, by (3) of Lemma 2.1, a normal 2-complement which is soluble by (1) of Lemma 2.1, so G is soluble contrary to the assumption. Thus, the subgroup H of G generated by all elements of order 3 is non-trivial.

If H is cyclic, then $C_G(H)$ contains a cyclic Sylow 3-subgroup which, clearly, lies in the center of its normalizer in $C_G(H)$. By (2) of Lemma 2.1, $C_G(H)$ contains a normal 3-complement which is soluble by previous paragraph. Since $|G : C_G(H)| \leqslant 2$, G is also soluble which is impossible.

Hence, by Theorem 1.1, H is isomorphic to $SL_2(3)$ or $SL_2(5)$. In particular, $|G : HC_G(H)| \leqslant 2$, so $G = HC_G(H)$. Since $|H \cap C_G(H)| = 2$, a Sylow 2-subgroup in $C_G(H)$ is cyclic, so $C_G(H)$ is soluble by (1) of Lemma 2.1 and, by assumption, $G = HC_G(H) = H$. The theorem is proved.

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